

Computing higher homotopy groups is $W[1]$ -hard

JIŘÍ MATOUŠEK*

Department of Applied Mathematics and
Institute of Theoretical Computer Science (ITI)
Charles University, Malostranské nám. 25
118 00 Praha 1, Czech Republic, and
Institute of Theoretical Computer Science
ETH Zurich, 8092 Zurich, Switzerland

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Abstract

Recently it was shown that, for every fixed $k \geq 2$, given a finite simply connected simplicial complex X , the k th homotopy group $\pi_k(X)$ can be computed in time polynomial in the number n of simplices of X . We prove that this problem is $W[1]$ -hard w.r.t. the parameter k even for X of dimension 4, and thus very unlikely to admit an algorithm with running time bound $f(k)n^C$ for an absolute constant C . We also simplify, by about 20 pages, a 1989 proof by Anick that, with k part of input, the computation of the rank of $\pi_k(X)$ is $\#P$ -hard.

Introduction. The homotopy groups $\pi_k(X)$, $k = 1, 2, \dots$, belong among the most important and most puzzling invariants of a topological space X (see, e.g., [Rav04, Koc90] for the amazing adventure of computing the homotopy groups of spheres, where only partial results have been obtained in spite of an enormous effort).¹

In this note we consider the (theoretical) complexity of computing $\pi_k(X)$, for given k and X . We assume that the space X is given as a finite simplicial complex, and the size of the input is measured as the number of simplices of X .

It is well known that the fundamental group $\pi_1(X)$ is uncomputable, as follows from undecidability of the word problem in groups [Nov55]. On the other hand, given a 1-connected X , i.e., one with $\pi_1(X)$ trivial, there are algorithms that compute $\pi_k(X)$, for every given $k \geq 2$ (more precisely, it is known that for a finite simplicial complex X , $\pi_k(X)$ is a finitely generated Abelian group, and the algorithms compute its isomorphism type, i.e., express it as a direct sum of cyclic groups). The first such algorithm is due to Brown [Bro57], and newer ones have been obtained as a part of general computational frameworks in algebraic topology due

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¹We recall that $\pi_k(X)$ defined as the set of all homotopy classes of *pointed* continuous maps $f: S^k \rightarrow Y$ (where S^k stands for the k -dimensional sphere), i.e., maps f that send a distinguished point $s_0 \in S^k$ to a distinguished point $x_0 \in X$. Here two pointed maps f, g are *homotopic* if f can be continuously deformed into g while keeping the image of s_0 fixed; this means that there is a continuous map $F: S^k \times [0, 1] \rightarrow X$ with $F(\cdot, 0) = f$, $F(\cdot, 1) = g$, and $F(s_0, \cdot) = x_0$. Strictly speaking, one should really write $\pi_k(X, x_0)$ instead of $\pi_k(X)$, but for a path-connected X , the choice of x_0 does not matter. Each $\pi_k(X)$, $k \geq 1$, is a group, which for $k \geq 2$ is Abelian, but the definition of the group operation is not important for us at the moment.

to Schön [Sch91] and due to and Sergeraert and his co-workers (e.g., [Ser94, RS12, Rea96]). Recently it was shown by Čadek et al. [ČKM⁺12] that, for every fixed $k \geq 2$, $\pi_k(X)$ can be computed in polynomial time, where the polynomial depends on k .

As for lower bounds, Anick [Ani89] proved that computing $\pi_k(X)$ is $\#P$ -hard, where X can even be assumed to be a 4-dimensional 1-connected space, but, crucially, k is regarded as a part of the input. The hardness also applies to the potentially easier problem of computing only the rank of $\pi_k(X)$, i.e., the number of direct summands isomorphic to \mathbb{Z} . In Anick’s original result, the space X is not given as a simplicial complex, but in another, considerably more compact representation, but it was shown by Čadek et al. [ČKM⁺13] that Anick’s representation can be converted into a simplicial complex with only polynomial-time overhead, and thus the $\#P$ -hardness result also applies to (1-connected, 4-dimensional, finite) simplicial complexes.

Results. Given that $\pi_k(X)$ is polynomial-time computable for k fixed, it is natural to ask whether it is *fixed-parameter tractable*, i.e., computable in time $f(k)n^C$ for some absolute constant C and some function f of k ; see, e.g., [Nie06] for an introduction to the field of *parameterized complexity*, which considers this kind of questions. We show that this is very unlikely.

Theorem 1. *The problem of computing $\pi_k(X)$ for a 4-dimensional 1-connected simplicial complex X with n simplices is $W[1]$ -hard (with parameter k).*

We refer to [Nie06] for the definition of the class of $W[1]$ -hard problems. Here it suffices to say that no problem in this class is known to be fixed-parameter tractable, and it is widely believed no $W[1]$ -hard problem is fixed-parameter tractable (this is somewhat similar to the widely held belief that $P \neq NP$).

The proof of Theorem 1 is very short and simple if we take two reductions from Anick [Ani89] for granted. At the same time, it gives a considerable simplification of Anick’s $\#P$ -hardness proof, replacing about 20 pages of Anick’s paper and a substantial part of its technical contents.

Vests. Anick [Ani89] defines an auxiliary computational problem called *vest* (“vector evaluated after a sequence of transformations”). The input instance \mathcal{V} of a vest is given by a (column) vector $\mathbf{v} \in \mathbb{Q}^d$, a list (T_1, T_2, \dots, T_m) of rational $d \times d$ matrices, and a $h \times d$ rational matrix S (for some natural numbers d, m, h). The M -sequence of such a \mathcal{V} is the integer sequence (M_1, M_2, \dots) , where

$$M_k := |\{(i_1, i_2, \dots, i_k) : ST_{i_k}T_{i_{k-1}} \cdots T_{i_1}\mathbf{v} = \mathbf{0}\}|,$$

with $\mathbf{0}$ denoting the (column) vector of h zeros.

Anick [Ani89] makes a connection of vests to the ranks of homotopy groups, which relies on other papers and apparently is not easy to trace down in detail. First, given an instance \mathcal{V} of a vest, one can construct a (suitable finite presentation of) a certain algebraic structure called a *123H-algebra* A , such that a suitable integer sequence associated with A (the *Tor-sequence* of A) equals the M -sequence of \mathcal{V} . This is stated as [Ani89, Thm. 3.4], but the proof refers to [Ani85, Thm. 1.3], which expresses the desired connection in a different language, and it is actually a special case of Theorem 7.6 of [Ani87].

Second, given the considered presentation of A , one can construct a 4-dimensional 1-connected cell complex X (which, in turn, can be converted into a simplicial complex in view of

[ČKM⁺13]) such that the Tor-sequence of A and the sequence of ranks $(\text{rk } \pi_2(X), \text{rk } \pi_3(X), \dots)$ are *rationally related*, which in particular means that the first k terms of one of the sequences can be computed from the first $O(k)$ terms of the other sequence, in polynomial time (with a *polynomial* dependence on k as well). The construction of X from A relies on Roos [Roo79].

It would be nice to streamline these reductions and have them summarized at one place, but here we take them for granted. In particular, they imply that $W[1]$ -hardness or $\#P$ -hardness of the vest problem implies $W[1]$ -hardness or $\#P$ -hardness of the problem of homotopy group computation considered in Theorem 1, respectively.

Hardness of vests: proof of Theorem 1. Given a graph G on n vertices, the problem of testing the existence of a clique (complete subgraph) on k vertices in G is one of the most famous and useful $W[1]$ -complete problems [Nie06].

For a given G and k , we construct a vest $\mathcal{V} = (\mathbf{v}, T_1, \dots, T_m, S)$ for which, with $s = k + \binom{k}{2}$, the s th term of the M -sequence is $M_s = s!C_k$, where C_k is the number of k -cliques in G .

Let us call a vector \mathbf{w} a *current vector* if it has the form $T_{i_j}T_{i_{j-1}} \cdots T_1\mathbf{v}$ for some j and some i_1, \dots, i_j . We will not describe the T_i explicitly as matrices; rather, we will say how T_i transforms the current vector \mathbf{w} into $T_i\mathbf{w}$, where we assume that all the components of \mathbf{w} that are not explicitly mentioned in such a description are left unchanged by T_i .

The initial vector \mathbf{v} , and thus all current vectors, have $d = n + 2m + 1$ components (where m , yet unspecified, is the number of the T_i). The first n components, called the *vertex components*, are in one-to-one correspondence with the vertices of G . Then there is a special component that equals 1 in \mathbf{v} , as well as in all current vectors (thus, no T_i is going to change it); all the other components of \mathbf{v} are set to 0. Finally, for each T_i , we have two *private components* in each current vector, which are changed by that T_i but by no other T_j .

Let a and b be the two private components belonging to some T_i ; we let T_i transform them to $a + 1$ and $b + a$, respectively (note that the 1 in $a + 1$ really means adding the special component to a). This guarantees that after at most one application of T_i , the second private component of T_i is 0, while two or more applications of T_i make it nonzero.

The T_i in \mathcal{V} are actually indexed by $V(G) \cup E(G)$, vertices and edges of the given graph (thus, $m = n + |E(G)|$). The T_i corresponding to a vertex v increments the vertex entry of v in the current vector by $k - 1$, while the T_i corresponding to an edge $\{u, v\}$ decrements the vertex entries of u and v each by 1.

It remains to specify the matrix S . We construct it as a zero-one matrix with a single 1 per row. Thus, the effect of multiplying the current vector by S is selecting certain components, and we construct S so that exactly the vertex components and the second private component of each T_i are selected; therefore, $h = n + m$.

Because of the private components, the vector $ST_{i_s}T_{i_{s-1}} \cdots T_1\mathbf{v}$ can be zero only if i_1, \dots, i_s are all distinct. Then it is easy to argue that exactly k vertex T_i 's and $\binom{k}{2}$ edge T_i 's must be used, corresponding to the vertex set and edge set of a k -clique in G , respectively. Since the ordering of such T_i 's is arbitrary, each k -clique contributes $s!$ to M_s . \square

The problem of counting k -cliques in a given graph, with k a part of input, is $\#P$ -complete, and thus the above proof also provides the promised simplification of Anick's $\#P$ -hardness proof.

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